Journal of Computational and Applied Mathematics 235 (2011) 4140–4148

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Precise and fast computation of a general incomplete elliptic integral of second kind by half and double argument transformations

Toshio Fukushima

National Astronomical Observatory of Japan, 2-21-1, Ohsawa, Mitaka, Tokyo 181-8588, Japan

ARTICLE INFO

Article history:
Received 17 December 2010
Received in revised form 27 February 2011

Keywords:
Incomplete elliptic integral
Jacobian elliptic function
Half argument transformation

ABSTRACT

We developed a method to compute simultaneously two associate incomplete elliptic integrals of the second kind, $B(\varphi|m)$ and $D(\varphi|m)$, by the half argument formulas of Jacobian elliptic functions and the double argument transformations of the integrals. The relative errors of the new method are sufficiently small as 5–10 machine epsilons. Meanwhile, the new method runs 3–6 times faster than that using Carlson's R_D. As a result, it enables a precise and fast computation of arbitrary linear combination of the incomplete elliptic integrals of the first and the second kind, $F(\varphi|m)$ and $E(\varphi|m)$.

1. Introduction

1.1. Incomplete elliptic integrals of first and second kind

The incomplete elliptic integrals of the first and the second kind are defined as

$$F(\varphi|m) \equiv \int_0^{\varphi} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}, \qquad E(\varphi|m) \equiv \int_0^{\varphi} \sqrt{1 - m\sin^2\theta}\,\mathrm{d}\theta. \tag{1}$$

Their textbooks are [1,2] and references are [3–7]. Also visit the website of [7]: http://dlmf.nist.gov/. The integrals appear in various fields of mathematical physics and engineering. Refer to Section 19.30 through Section 19.35 of [7].

Originally, the elliptic integrals are discussed in computing the lengths of basic plane curves: $F(\varphi|m)$ for Bernoulli's lemniscate and $E(\varphi|m)$ for an ellipse [8]. The square root form of the integrand provides their usefulness in describing the physical quantities under the influence of forces of the inverse-square-law type: Newton's classic gravitation and Coulomb's electrostatic force. For example, the electrostatic potential of a conducting triaxial ellipsoid is described in terms of $F(\varphi|m)$ [9]. Similar expressions in case of the gravitational potential serve as a good model of those of galaxies [10]. Also $F(\varphi|m)$ appears in the Schwarz–Christoffel mapping from the square to the upper half plane [11].

Among others, the most important feature of $F(\varphi|m)$ is its tight relation with Jacobian elliptic functions, $\mathrm{sn}(u|m)$, $\mathrm{cn}(u|m)$, and $\mathrm{dn}(u|m)$ [12–17]. In fact, the integral is also interpreted as their inverse in the sense that $\mathrm{sn}(u|m) = \sin\varphi$ and $\mathrm{cn}(u|m) = \cos\varphi$ if $u = F(\varphi|m)$. See Chapter 22 of [7]. The elliptic functions as well as the elliptic integrals are essentially needed in constructing the dynamical theories of the rotational motion of rigid body [18–20]. These are not only important in astronomy but also in physics and chemistry, especially in the symplectic integration of the rotational motion of molecules. See [21] and the references therein.

On the other hand, $E(\varphi|m)$ represents the length of meridional arc of a spheroid. In this case, the amplitude φ means the geodetic latitude while the parameter m does the square of the first eccentricity of the spheroid [22]. This expression still plays a key role in geodesy and cartography [23]. Indeed, the integral is the base to develop the Gauss–Krüger projection method, the most popular method of mapping [24].

E-mail address: Toshio.Fukushima@nao.ac.jp.

0377-0427/$ – see front matter
doi:10.1016/j.cam.2011.03.004

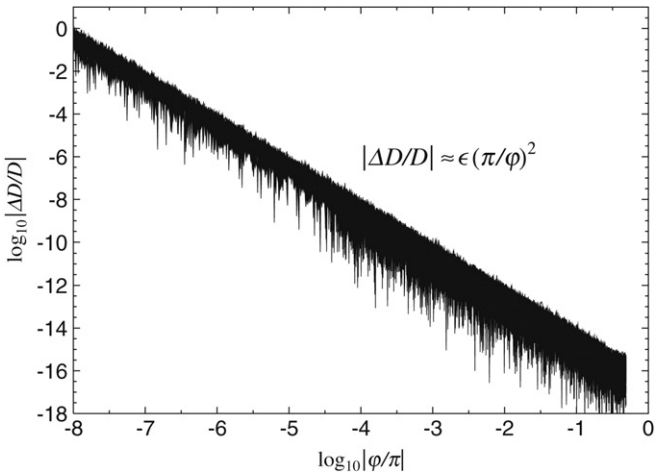

Fig. 1. Round-off errors of $D(\varphi|m)$ computed by `el2`. Shown are the relative errors of $D(\varphi|m)$ computed by Bulirsch's `el2` in the double precision environment. The errors are measured as the differences from the results of the quadruple precision computation and plotted as functions of φ in a log–log manner. Overlapped are the error curves with various values of m in the standard domain, $0 < m < 1$.

1.2. Associate incomplete elliptic integrals of second kind

In many applications, $F(\varphi|m)$ and $E(\varphi|m)$ are used in combination [25]. In case of the lengths of plane curves, that of hyperbola requires both two as shown in the introduction of [4]. Also some inductance problems demand solutions in a linear combination of these two integrals [9]. Further, both of them are needed in computing their partial derivatives with respect to m [7]. These are necessary in computing the force field of potentials expressed using $F(\varphi|m)$. Similar needs arise in evaluating the partial derivatives of Jacobian elliptic functions with respect to u and/or m [4].

From a practical viewpoint, however, rather important is not the pair of $F(\varphi|m)$ and $E(\varphi|m)$ but a pair of associate incomplete elliptic integrals of the second kind [26]:

$$B(\varphi|m) \equiv \int_0^\varphi \frac{\cos^2\theta\,\mathrm{d}\theta}{\sqrt{1-m\sin^2\theta}}, \qquad D(\varphi|m) \equiv \int_0^\varphi \frac{\sin^2\theta\,\mathrm{d}\theta}{\sqrt{1-m\sin^2\theta}}. \tag{2}$$

Once they are given, $F(\varphi|m)$ and $E(\varphi|m)$ are computed without loss of precision as

$$F(\varphi|m) = B(\varphi|m) + D(\varphi|m), \qquad E(\varphi|m) = B(\varphi|m) + m_c D(\varphi|m), \tag{3}$$

where $m_c \equiv 1 - m$ is the complementary parameter. Meanwhile, the reverse transformation from $F(\varphi|m)$ and $E(\varphi|m)$ to $B(\varphi|m)$ and $D(\varphi|m)$ suffers significant round-off errors when $m \approx 0$.

1.3. Existing formulations to compute incomplete elliptic integrals

The existing methods to compute the incomplete elliptic integrals are classified into three categories: the series expansion formulas [3,27,28,4,29,30], the methods based on Landen transformations [31,26,32–37], and the formulations using the duplication theorems [38,39,16,40].

Noting the importance of the associate incomplete elliptic integrals, Bulirsch designed `el2` [26] to compute an arbitrary linear combination of them as

$$\texttt{el2}(t, k_c, a, b) \equiv aB(\varphi|m) + bD(\varphi|m), \tag{4}$$

where $t \equiv \tan\varphi$ and $k_c \equiv \sqrt{m_c}$ is the complementary modulus. Unfortunately, his algorithm based on the descending Landen transform faces a severe information loss in computing $D(\varphi|m)$ when φ is small [35]. See Fig. 1 showing that the relative errors of $D(\varphi|m)$ computed by `el2` are roughly in proportion to $1/\varphi^2$ independently on m. The reason will be explained in Appendix A.

On the other hand, Carlson reconstructed the theory of elliptic integrals by introducing their symmetric forms [41–43,38,44–48,16,49]. See Chapter 19 of [7]. Within his framework, the associate integrals are effectively expressed by the symmetric elliptic integral of the second kind

$$R_D(x, y, z) \equiv \frac{3}{2}\int_0^\infty \frac{\mathrm{d}t}{\sqrt{(t+x)(t+y)(t+z)^3}}, \tag{5}$$

as

$$B(\varphi|m) = \frac{1}{3}m_c s^3 R_D(c^2, 1, d^2) + \frac{sc}{d}, \qquad D(\varphi|m) = \frac{1}{3}s^3 R_D(c^2, d^2, 1), \tag{6}$$

where $s \equiv \sin\varphi$, $c \equiv \cos\varphi$, and $d \equiv \sqrt{1 - ms^2}$. Refer to Section 19.25 of [7].

No cancelation problems occur here as Carlson stressed. However, the numerical evaluation of R_D by his duplication method [38,39,16] consumes a considerable amount of computational time. In fact, the CPU time of its improved algorithm [16,40] is still around 20 times more than that of the sine function in the double precision environment. See Table 1 presented later. This is a significant computational labor if considering its frequent usage.

1.4. Outline of the present article

Recently, we developed a fast method to compute $F(\varphi|m)$ in [50]. Hereafter we call it Paper I. It constitutes a part of our efforts to accelerate the procedures to compute the complete and incomplete elliptic integrals and the Jacobian elliptic functions [37,51–53,30,54].

The key technique we used in Paper I is the combination of the half argument formulas of $\mathrm{sn}(u|m)$ and $\mathrm{cn}(u|m)$ and the Maclaurin series expansion of $F(\varphi|m)$. The new method is sufficiently precise as the existing procedures and yet significantly faster than them. Paper I shows that our routine to compute $F(\varphi|m)$ is 1.2–1.6 times faster than Bulirsch's `el1` and 1.9–2.2 times faster than Carlson's R_F.

In this article, we report its adaptation to the simultaneous computation of $B(\varphi|m)$ and $D(\varphi|m)$. First, we explain the adapted method in Section 2. Then, we compare the cost and performance of the new method with those of the existing procedures in Section 3.

2. Method

2.1. Selection rule

We assume that φ and m are reduced such that $0 < \varphi < \pi/2$ and $0 < m < 1$ throughout this section. Refer to Appendix B for a practical procedure to realize this condition. Following Paper I, we select one of the four expressions depending on the values of φ and m as

$$B(\varphi|m) = \begin{cases} B_s(s|m), & (\text{if } \varphi < \varphi_S) \\ B(m) - B_s(z|m) + sz, & (\text{elseif } z^2 < y_S) \\ B_c(c|m), & (\text{elseif } c < w) \\ B(m) - B_c(w|m) + \sqrt{(1-c^2)(1-w^2)}, & (\text{otherwise}) \end{cases} \tag{7}$$

$$D(\varphi|m) = \begin{cases} D_s(s|m), & (\text{if } \varphi < \varphi_S) \\ D(m) - D_s(z|m) - sz, & (\text{elseif } z^2 < y_S) \\ D_c(c|m), & (\text{elseif } c < w) \\ D(m) - D_c(w|m) - \sqrt{(1-c^2)(1-w^2)}, & (\text{otherwise}) \end{cases} \tag{8}$$

where $B(m) \equiv B(\pi/2|m)$ and $D(m) \equiv D(\pi/2|m)$ are the corresponding complete integrals,

$$B_s(s|m) \equiv B(\sin^{-1}s|m), \qquad B_c(c|m) \equiv B(\cos^{-1}c|m), \tag{9}$$

$$D_s(s|m) \equiv D(\sin^{-1}s|m), \qquad D_c(c|m) \equiv D(\cos^{-1}c|m), \tag{10}$$

are the abbreviations of the integrals with different forms of the first input argument:

$$s \equiv \sin\varphi, \qquad c \equiv \cos\varphi, \qquad z \equiv \frac{c}{\sqrt{m_c + mc^2}}, \qquad w \equiv \sqrt{1 - z^2} = \sqrt{\frac{m_c}{1 - ms^2}}\,s. \tag{11}$$

Meanwhile, φ_S and y_S are constants to govern the above selection rules as

$$y_S \equiv 0.9, \quad \varphi_S \equiv 1.249 \approx \sin^{-1}\sqrt{y_S}. \tag{12}$$

Refer to Paper I for the discussion to adopt these values. An efficient procedure to compute $B(m)$ and $D(m)$ is found in [30]. The second and the fourth expression in (7) and (8) are the rewritings of the first and the third one by means of the special addition formulas of $F(\varphi|m)$ and $E(\varphi|m)$ obtained from formula 117.01 of [4].

2.2. Half and double argument transformations

The computation of $B_s(s|m)$ and $D_s(s|m)$ is the core part of this formulation. We conduct it by (1) successive applications of the half argument transformation with respect to $y \equiv s^2$ so as to decrease y while keeping m the same, (2) the evaluation

of the integrals for the decreased y by their truncated Maclaurin series expansions, and (3) the recovery of the integral values for the original s by repeating the application of the double argument transformation of the integrals. Similarly we calculate $B_c(c|m)$ and $D_c(c|m)$ by (1) applying successively the half argument transformation with respect to $x \equiv c^2$ so as to increase x while keeping m the same, (2) calling the routine to compute $B_s(s|m)$ and $D_s(s|m)$ described in the above when the increased x is sufficiently large, and (3) obtaining the integral values for the original c by repeating the application of the double argument transformation.

Let us be more specific. Consider to evaluate the associate integrals, $B_0 \equiv B_s(s_0|m)$ and $D_0 \equiv D_s(s_0|m)$. We construct a sequence of y by starting from $y_0 \equiv s_0^2$ and applying successively the half argument transformation to them as

$$y_{j+1} = \frac{y_j}{(1+c_j)(1+d_j)}, \quad (j = 0, 1, \ldots), \tag{13}$$

where $c_j \equiv \sqrt{1-y_j}$ and $d_j \equiv \sqrt{1-my_j}$. This is a rewriting of the half argument formula of $\mathrm{sn}(u|m)$ given in the first part of formula 124.02 of [4]. It provides a decreasing sequence of y_j since the divisor $(1+c_j)(1+d_j)$ is greater than unity; refer to Paper I. The intermediate values, y_j, are stored for later use. The sequence of the transformation is terminated when

$$y_j < y_A(m) \equiv \begin{cases} 0.04094 - 0.00652m, & \text{(double precision)} \\ 0.1888 - 0.0378m, & \text{(single precision).} \end{cases} \tag{14}$$

Refer to Paper I on the determination of the conditions.

Denote by J the index at the termination. Then we approximate the corresponding integral values, $B_J \equiv B_s(s_J|m)$ and $D_J \equiv D_s(s_J|m)$, by their truncated Maclaurin series expansions as

$$B_J \approx s_J \sum_{\ell=0}^{L} B_\ell(m) y_J^\ell, \qquad D_J \approx s_J \sum_{\ell=1}^{L} D_\ell(m) y_J^\ell, \tag{15}$$

where $s_J \equiv \sqrt{y_J}$ and the order of approximate polynomials, L, is 6 and 9 in the single and double precision environments, respectively. Refer to Paper I on this choice of the optimal orders. The computation of the expansion coefficients, $B_\ell(m)$ and $D_\ell(m)$, will be described in Section 2.3 later.

Finally, we obtain B_0 and D_0 by conducting the double argument transformation of $B(\varphi|m)$ and $D(\varphi|m)$ repeatedly as

$$B_{j-1} = 2B_j - s_{j-1}y_j, \qquad D_{j-1} = 2D_j + s_{j-1}y_j, \quad (j = J, J-1, \ldots, 1), \tag{16}$$

where $s_j \equiv \sqrt{y_j}$. These formulas are derived from the addition theorems of $F(\varphi|m)$ and $E(\varphi|m)$ given in formula 116.01 of [4].

The above algorithm faces cancelation problems when y_0 and/or m are close to unity. This occurs in the computation of small factors, c_0 and/or d_0. In that case, we use another main variable, $x \equiv 1-y$, and replace the half argument transformation of y with that of x:

$$x_{j+1} = \frac{c_j + d_j}{1 + d_j}, \quad (j = 0, 1, \ldots) \tag{17}$$

where $c_j \equiv \sqrt{x_j}$ and $d_j \equiv \sqrt{m_c + mx_j}$ this time. This is a rewriting of the half argument formula of $\mathrm{cn}(u|m)$ given in the second part of formula 124.02 of [4]. It leads to an increasing sequence of x_j. If x_j becomes sufficiently large by repeated usage of this transformation, we shift to y_j by the translation $y_j = 1 - x_j$. We set $x_S \equiv 1 - y_S = 0.1$ as the critical value of x_j to shift to the computation in terms of y_j. A dangerous divisor m appears nowhere in the above procedure. This ensures the robustness of the present algorithm against small values of m.

2.3. Maclaurin series expansions

Once y becomes sufficiently small, we evaluate the associate integrals by their Maclaurin series expansions with respect to $s \equiv \sqrt{y}$. The expansion coefficients, $B_\ell(m)$ and $D_\ell(m)$, are ℓth order polynomials of m. We will not discuss their general expressions immediately. Instead, we consider the Maclaurin series expansion of $F_s(s|m) \equiv F(\sin^{-1} s|m)$ given in Paper I:

$$F_s(s|m) = s \sum_{\ell=0}^{\infty} F_\ell(m) y^\ell, \tag{18}$$

where $F_0(m) = 1$ and $F_\ell(m)$ is another ℓth order polynomial of m as

$$F_\ell(m) = \frac{1}{2^\ell(2\ell+1)} \sum_{j=0}^{\ell} \left(\frac{(2j-1)!!(2\ell-2j-1)!!}{j!(\ell-j)!} \right) m^j. \quad (\ell = 1, 2, \ldots). \tag{19}$$

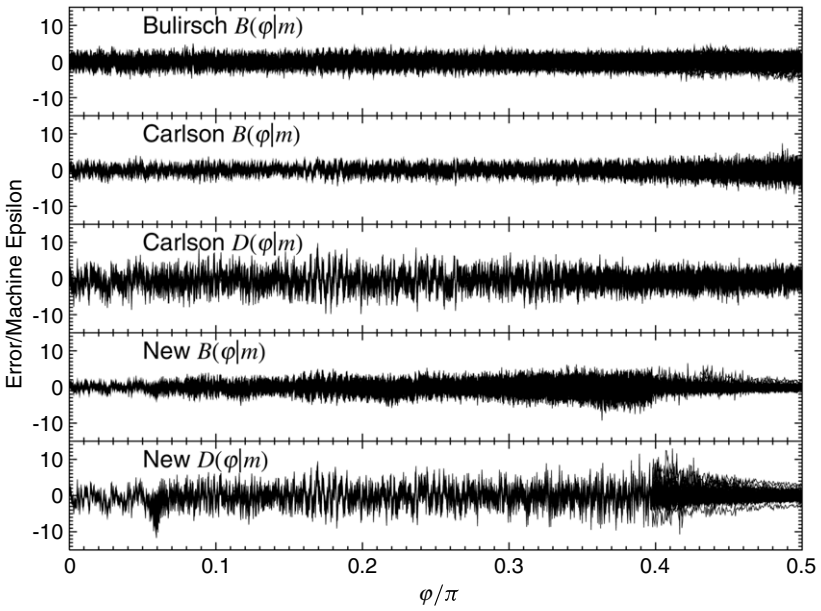

Fig. 2. Amplitude Dependence of Relative Errors of Associate Integrals. Shown are the relative errors of $B(\varphi|m)$ and $D(\varphi|m)$ in the double precision environment. Compared methods are (1) Bulirsch's `el2` [26], (2) Carlson's R_D [38], and (3) `elbd`, the new procedure explained in Section 2 of the main text. A change of manner of errors around $\varphi/\pi \sim 0.4$ in the new method is due to the switch structure in its base algorithm shown in (7) and (8).

Following the approach in Paper I to obtain $F_\ell(m)$, we similarly obtain the expressions of $D_\ell(m)$ as

$$D_\ell(m) \equiv \frac{1}{2^{\ell-1}(2\ell+1)} \sum_{j=0}^{\ell-1} \left(\frac{(2j-1)!!(2\ell-2j-3)!!}{j!(\ell-j-1)!} \right) m^j, \quad (\ell = 1, 2, \ldots) \tag{20}$$

and $D_0(m) = 0$. Comparing these, we learn that $D_\ell(m)$ is computable from $F_\ell(m)$ as

$$D_\ell(m) = \left(\frac{2\ell-1}{2\ell+1} \right) F_{\ell-1}(m). \quad (\ell = 1, 2, \ldots). \tag{21}$$

On the other hand, the relation among $B(\varphi|m)$, $D(\varphi|m)$, and $F(\varphi|m)$ is expressed as

$$B(\varphi|m) = F(\varphi|m) - D(\varphi|m). \tag{22}$$

This leads to a similar relation among their expansion coefficients as

$$B_\ell(m) = F_\ell(m) - D_\ell(m). \quad (\ell = 0, 1, \ldots). \tag{23}$$

As a result, $B_0(m) = 1$.

Now the preparation is completed. Our recipe is (1) to compute $F_\ell(m)$ by (19), (2) to evaluate $D_\ell(m)$ as (21), and (3) to obtain $B_\ell(m)$ from (23). This formulation significantly accelerates the simultaneous evaluation of $B_s(s|m)$ and $D_s(s|m)$. In fact, the separate computation of $B_\ell(m)$ and $D_\ell(m)$ by their polynomial expressions would require two double summations. Meanwhile, the above procedure requires only one double summation in computing $F_\ell(m)$. Therefore, the computation time of the approximate polynomials is roughly halved by the introduction of this technique.

3. Comparison with existing methods

3.1. Computational precision

Let us compare the computational cost and performance of the new and existing methods explained in Section 1. First, we investigate the computational errors. Fig. 2 shows the relative errors of the new and existing procedures in computing $B(\varphi|m)$ and $D(\varphi|m)$. It depicts the errors in the double precision environment as functions of φ for various values of m. The errors are (1) measured as the differences from the quadruple precision results obtained by `qel2`, the quadruple precision extension of `el2`, (2) normalized by the magnitude of the integrals, (3) scaled by the machine epsilon, and (4) illustrated as functions of φ for $0 < \varphi < \pi/2$. We confirmed that the errors of `qel2` are far less than the double precision machine epsilon despite its cancelation problem. Superposed are the curves with various values of m in its standard domain, namely for the cases $m = 0.01, 0.02, \ldots, 0.99$. Compared methods are (1) Bulirsch's `el2`, (2) $D(\varphi|m)$ by `rd`, the improved version

Table 1
Averaged CPU Times to Compute $B(\varphi|m)$ and $D(\varphi|m)$. Listed are the averaged CPU times of the new and existing methods to compute $B(\varphi|m)$ and $D(\varphi|m)$ simultaneously for the case $0 < m < 1$. The unit of CPU time is that to compute the sine function in the double precision environment.

Method	Procedures	Reference	$0 < \varphi < \pi/2$		$-\infty < \varphi < \infty$	
			Single	Double	Single	Double
Duplication	`rd`	[16,40]	22.7	33.4	55.1	85.1
Hybrid	`el2`, `rd`, `cel`	[26,16,40]	24.7	29.7	36.8	60.1
New	`elbd`, `celbd`	This article, [30]	7.4	10.9	9.3	17.9

to compute R_D by the duplication method, and (3) our `elbd`, a procedure based on the new algorithm explained in the previous section. We omit the errors of $D(\varphi|m)$ computed by `el2` here since they would be scaled out; refer to Fig. 1. At any rate, we conclude that all the errors are satisfactorily small except the case of $D(\varphi|m)$ by `el2`.

3.2. Computational speed

Table 1 compares the CPU times of the new and existing methods to compute the pair of $B(\varphi|m)$ and $D(\varphi|m)$ and the quartet of $B(\varphi|m)$, $D(\varphi|m)$, $B(m)$, and $D(m)$ in the single and double precision environments, respectively. The complete integrals, $B(m)$ and $D(m)$, are required in calculating the incomplete integrals, $B(\varphi|m)$ and $D(\varphi|m)$, for arbitrary value of φ; refer to Appendix B. Thus the latter comparison is the more practical in this sense.

We experimentally learn that Bulirsch's routines run significantly faster than the corresponding Carlson's procedures. Meanwhile, `el2` is imprecise in computing $D(\varphi|m)$ due to the cancelation problem. Then, we combined Bulirsch's and Carlson's methods in order to construct the fastest *and* precise method using the existing procedures. The hybrid method computes (1) $B(\varphi|m)$ by `el2`, (2) $D(\varphi|m)$ by `rd`, and (3) $B(m)$ and $D(m)$ by Bulirsch's `cel` [26]. Meanwhile, the new method computes the incomplete integrals by `elbd` and the complete integrals by `celbd` described in [30]. All the procedures to compute the complete integrals are sufficiently precise as illustrated in [52,53,30].

The listed CPU times are those uniformly averaged with respect to φ and m and normalized by that to compute the sine function in the double precision environment. Actually, we measured the CPU times as simple means of the results for 4095×4095 equally spaced grid points in the given domains of φ and m. We fixed the latter as $0 < m < 1$ while changing the former as $0 < \varphi < \pi/2$ and $|\varphi| < 1000$, respectively. All the computation codes were (1) written in Fortran 77/90, (2) compiled by the Intel Visual Fortran 8.0, and (3) executed at a PC with an Intel Core Duo processor under Windows XP. The table clearly shows that the new method runs roughly 3–6 times faster than the existing procedures. This mainly owes to the simplicity of the half and double argument transformations and the effectiveness of the adopted selection rule.

4. Conclusion

We created a new method to calculate simultaneously two associate incomplete elliptic integrals of the second kind, $B(\varphi|m)$ and $D(\varphi|m)$. This was done by adopting the same approach we took in developing a fast method to compute $F(\varphi|m)$ in our recent work [50]. The main technique consists of three parts: (1) the half argument formulas of $\mathrm{sn}(u|m)$ and/or $\mathrm{cn}(u|m)$, (2) the truncated Maclaurin series expansion of $B(\varphi|m)$ and $D(\varphi|m)$ with respect to $\sin\varphi$, and (3) the double argument transformations of $B(\varphi|m)$ and $D(\varphi|m)$. The formulas in the new method contain no small divisor such as m. Thus it is robust against the small values of m. In order to accelerate the computation in case $\varphi \approx \pi/2$, we designed the new method to use the associate complete elliptic integrals, $B(m)$ and $D(m)$. Their fast computation was given in another recent work of ours [30]. This trick significantly reduces the round-off errors in the so-called critical region where $\varphi \approx \pi/2$ and $m \approx 1$.

As a result, the new method is sufficiently precise such that the resulting relative errors are less than 10 machine epsilons in the double precision environment as illustrated in Fig. 2. We confirmed that the magnitude of maximum error reduces to around 5 machine epsilons in the single precision environment. Meanwhile, the new method runs significantly faster than the existing procedures as shown in Table 1. Its CPU time is 3.1–5.9 times smaller than that of Carlson's R_D when $|\varphi| < \pi/2$. In the case of arbitrary values of φ, the CPU time ratio slightly changes to 3.1–4.8 if we combine the new method with our method to compute complete integrals.

In conclusion, the new method offers a precise and efficient way to evaluate arbitrary linear combination of $F(\varphi|m)$ and $E(\varphi|m)$. The Fortran programs of `elbd` and `celbd` are available from the author upon request.

Acknowledgments

The author thanks the anonymous referees for their valuable advice.

Appendix A. Reason of precision loss of Bulirsch's `el2`

Bulirsch's routine `el2` [26] is known to face cancelation problems for small values of $|\varphi|$ [35]. This happens for certain combinations of linear coefficients, a and b, and occurs independently on the value of m. See Fig. 1 again for the typical case,

$D(\varphi|m)$. A careful examination of the published algorithm of `el2` reveals that it is computed as the sum of two components as

$$\texttt{el2}\,(t, k_c, a, b) = \left(\frac{aB(m) + bD(m)}{K(m)}\right) F(\varphi|m) + \left(\frac{a-b}{m}\right) Z(\varphi|m). \tag{A.1}$$

Here $K(m) = B(m) + D(m)$ is the complete elliptic integral of the first kind and $Z(\varphi|m)$ is the Jacobian zeta function [4] defined as

$$Z(\varphi|m) \equiv E(\varphi|m) - \left(\frac{E(m)}{K(m)}\right) F(\varphi|m), \tag{A.2}$$

where $E(m) = B(m) + m_c D(m)$ is the complete elliptic integral of the second kind.

Assume that φ and m are in their standard domain, $0 < \varphi < \pi/2$ and $0 < m < 1$. Then, $K(m)$, $B(m)$, $D(m)$, $F(\varphi|m)$, and $Z(\varphi|m)$ are all positive definite. Thus the two components in (A.1) have the opposite signs if $0 \le a < b$ or $b < a \le 0$. This would cause a cancelation. Consider an extreme case when $a = 0$ and $b = 1$. This corresponds to $D(\varphi|m)$. Then, the expression of `el2` becomes the difference of similar quantities as

$$\texttt{el2}(t, k_c, 0, 1) = \left(\frac{D(m)B(\varphi|m) + D(m)D(\varphi|m)}{K(m)}\right) - \left(\frac{D(m)B(\varphi|m) - B(m)D(\varphi|m)}{K(m)}\right). \tag{A.3}$$

Both of the two terms on the right-hand side share a factor, $D(m)B(\varphi|m)/K(m)$. This factor becomes dominant independently on m when φ is small. This is because the main terms of Maclaurin series expansion of $B(\varphi|m)$ and $D(\varphi|m)$ with respect to $s \equiv \sin\varphi$ are of the different orders as

$$B(\varphi|m) = s + \frac{m_c s^3}{6} + \cdots, \qquad D(\varphi|m) = \frac{s^3}{3} + \cdots. \tag{A.4}$$

Therefore, `el2` contains a source of cancelation when $|\varphi|$ is small for whatever value of m.

Appendix B. Reduction of input arguments

The associate incomplete elliptic integrals of the second kind, $B(\varphi|m)$ and $D(\varphi|m)$, are real-valued for arbitrary real φ if $m < 1$ and for $|\varphi| < \sin^{-1}(1/\sqrt{m})$ if $m \ge 1$. Let us reduce φ and m such that $0 < \varphi < \pi/2$ and $0 < m < 1$. First, in Appendix B.1, we show a series of formulas to reduce φ such that $0 < \varphi < \pi/2$ for arbitrary real m. Then, in Appendix B.2, we provide a group of formulas to reduce m such that $0 < m < 1$ for φ in the standard domain, $(0, \pi/2)$, and satisfying the condition, $m\sin^2\varphi < 1$, if $m \ge 1$.

B.1. Reduction of amplitude

We begin with the reduction of φ. If $\varphi = 0$ or $\varphi = \pi/2$, the integrals are expressed as

$$B(0|m) = D(0|m) = 0, \qquad B(\pi/2|m) = B(m), \qquad D(\pi/2|m) = D(m), \tag{B.1}$$

where $B(m)$ and $D(m)$ are the corresponding complete elliptic integrals. Their fast computation is explained in [30].

Next we assume that $|\varphi| > \pi/2$. This implies that $m < 1$. Then we reduce φ so as to lie in $(-\pi/2, \pi/2)$ by utilizing the amplitude modulus transformation:

$$B(\varphi|m) = 2jB(m) + B(\varphi - j\pi|m), \qquad D(\varphi|m) = 2jD(m) + D(\varphi - j\pi|m), \tag{B.2}$$

where j is an integer such that $|\varphi - j\pi| < \pi/2$. These are derived from those of $F(\varphi|m)$ and $E(\varphi|m)$ given as formulas 113.01 and 113.02 of [4]. When $m < 0$, the complete integrals are computed from those of positive parameters by the negative parameter transformation:

$$B(m) = \sqrt{1 - m_N}\, D(m_N), \qquad D(m) = \sqrt{1 - m_N}\, B(m_N), \tag{B.3}$$

where

$$m_N \equiv \frac{-m}{1-m}, \tag{B.4}$$

is the transformed parameter. These are rewritings of those of $K(m)$ and $E(m)$ derived from the first and second parts of formula 160.02 of [4]. When $m < 0$, the transformed parameter lies in the standard domain as $0 < m_N < 1$.

Finally, if $\varphi < 0$, we make φ positive by the negative amplitude transformation:

$$B(\varphi|m) = -B(-\varphi|m), \qquad D(\varphi|m) = -D(-\varphi|m). \tag{B.5}$$

Thus we reduce φ such that $0 < \varphi < \pi/2$ for arbitrary m.

B.2. Reduction of parameter

Let us move to the reduction of m under the conditions $0 < \varphi < \pi/2$ and $m \sin^2 \varphi < 1$. First, if $m = 0$ or $m = 1$, the integrals are expressed by elementary functions as

$$B(\varphi|0) = \frac{\varphi}{2} + \frac{\sin 2\varphi}{4}, \qquad D(\varphi|0) = \frac{\varphi}{2} - \frac{\sin 2\varphi}{4}, \tag{B.6}$$

$$B(\varphi|1) = \sin\varphi, \qquad D(\varphi|1) = \log\left(\frac{1+\sin\varphi}{\cos\varphi}\right) - \sin\varphi. \tag{B.7}$$

These expressions are obtained from those of $F(\varphi|m)$ and $E(\varphi|m)$ given as the second and first parts of formulas 111.01 and 111.04 of [4], respectively.

Next, we assume that $m > 1$. This means that φ is in a limited range as $0 < \varphi < \sin^{-1}(1/\sqrt{m})$. Then, we reduce m so as to lie in the standard domain, (0, 1), by the reciprocal parameter transformation:

$$B(\varphi|m) = \sqrt{m_R}[B(\varphi_R|m_R) + (1 - m_R)D(\varphi_R|m_R)], \tag{B.8}$$

$$D(\varphi|m) = m_R\sqrt{m_R}D(\varphi_R|m_R), \tag{B.9}$$

where

$$\varphi_R \equiv \sin^{-1}(\sqrt{m}\ \sin\varphi), \qquad m_R \equiv \frac{1}{m}, \tag{B.10}$$

are the transformed input arguments. These are obtained from the transformation formulas of $F(\varphi|m)$ and $E(\varphi|m)$ given as the second and first parts of formula 114.01 of [4]. Note that $0 < m_R < 1$ because $m > 1$. Also φ_R is uniquely determined and real-valued since $0 < \sqrt{m}\sin\varphi < 1$.

Third, we assume that $m < 0$. Then, we make m positive by the negative parameter transformation:

$$B(\varphi|m) = \sqrt{1 - m_N}\left(D(\varphi_N|m_N) + \frac{\sin\varphi_N\cos\varphi_N}{\sqrt{1 - m_N\sin^2\varphi_N}}\right), \tag{B.11}$$

$$D(\varphi|m) = \sqrt{1 - m_N}\left(B(\varphi_N|m_N) - \frac{\sin\varphi_N\cos\varphi_N}{\sqrt{1 - m_N\sin^2\varphi_N}}\right), \tag{B.12}$$

where

$$\varphi_N \equiv \sin^{-1}\left(\sqrt{\frac{m_c}{1 - m\sin^2\varphi}}\ \sin\varphi\right), \tag{B.13}$$

is the transformed amplitude and m_N is already introduced in (B.4). These are obtained from those of $F(\varphi|m)$ and $E(\varphi|m)$ given as the first and second parts of formula 160.02 of [4]. Note that $m < 0$ and $0 < \varphi < \pi/2$. Then $m_c \sin^2\varphi < 1 - m\sin^2\varphi$. Thus, the argument of the inverse sine function in (B.13) is less than unity. Therefore, φ_N is uniquely determined and real-valued. In conclusion, we reduce m such that $0 < m < 1$.

B.3. Comment

Finally, we add a comment that our choice of the main variable, $y \equiv \sin^2\varphi$, decreases the computational labor of the transformation significantly in the last two cases. In fact, we translate the transformations in terms of φ into those in terms of y as

$$y_R = my, \qquad y_N = \frac{m_c y}{1 - my}. \tag{B.14}$$

Also the additional combination of trigonometric functions are expressed in terms of y as

$$\frac{\sin\varphi_N\cos\varphi_N}{\sqrt{1 - m_N\sin^2\varphi_N}} = \sqrt{\frac{y_N(1 - y_N)}{1 - m_N y_N}}. \tag{B.15}$$

Therefore, the computational labor of the transformations mainly consists of one or two calls of the square root.

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